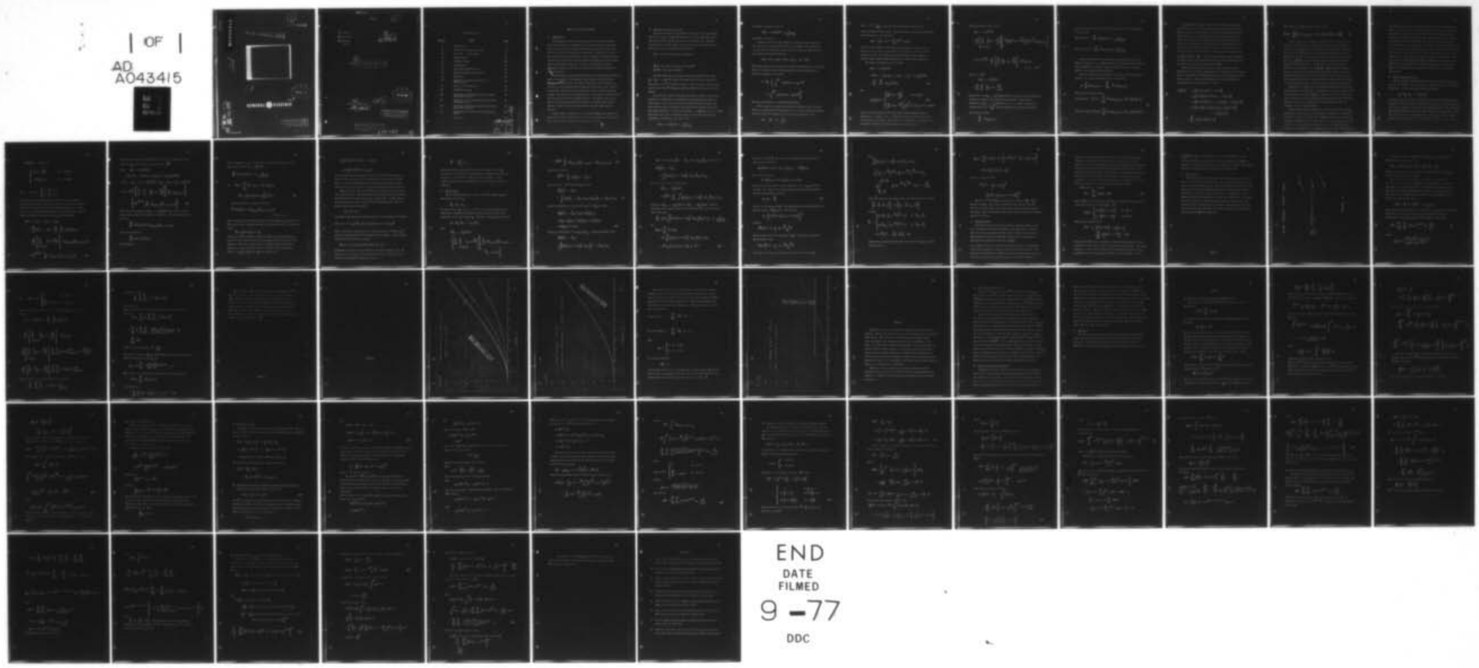
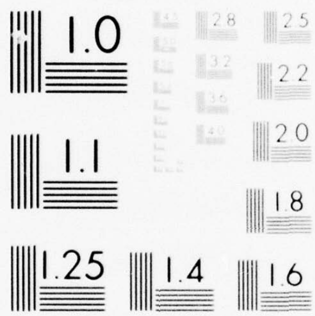
AD-A043 415 GENERAL ELECTRIC CO SYRACUSE N Y HEAVY MILITARY ELEC--ETC F/G 17/1
MULTIPLE PING SONAR DETECTION.(U)
SEP 66 E HERMANNS
UNCLASSIFIED TM-341-276 NL

| OF |

AD
A043415



END
DATE
FILMED
9-77
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

ADA043415

MOST Project -3

GENERAL ELECTRIC

FN-412-D (12-50)

1

(1)

ADVANCED
ANALYTICAL
METHODS
UNIT

(9) TECHNICAL MEMO.
(14) TM-341-276

(6)
TITLE: MULTIPLE PING SONAR DETECTION.

(10)
AUTHOR: E. Hermanns

(12) 61p.

(11)
DATE: Sept 1966

GENERAL ELECTRIC
HEAVY MILITARY ELECTRONICS DEPARTMENT
SYRACUSE, NEW YORK

149 510

LB

TABLE OF CONTENTS

MULTIPLE PING SONAR DETECTION

1. Introduction

The design of detectors for single ping sonar has closely followed radar design principles. Matched filters, optimum only for a stationary, Gaussian noise channel, are used for detection although ocean noise has been shown to be neither Gaussian nor stationary. Although much has been learned about the sonar environment in recent studies, more accurate design and analysis of detectors must await detailed knowledge of the ocean medium which is unavailable at present. The radar analysis and the design principles derived therefrom are thus applied, essentially unchanged, to single ping sonar.

In the design of multiple ping sonar there is an additional complication which is not present in the analogous radar problem studied by Marcum ~~(Reference 1)~~. In the radar case all of the target's characteristics are assumed to remain stationary for the duration of the pulse train. This is a reasonable assumption since the entire pulse train may last only microseconds. But in the 20 to 30 seconds between sonar pings the target location and other characteristics can change markedly. The problem, then, is to detect the presence and estimate the location of a target whose path of motion is subject to certain constraints but otherwise unknown to the observer.

In this report a formulation of the optimum multiple ping detector is presented which takes into account the stationarity differences between the sonar problem and the ideal radar formulation by Marcum.

2. Derivation of Optimal Processor

Let N be the total number of (discrete) locations under surveillance by the sonar system. By the j^{th} ping there are N^j possible paths which the target may have traveled through the surveillance area. The data available from the last j pings is to be processed in an optimal fashion to decide between the following $1+N^j$ hypotheses:

H(1): the target is present with path #1

.

.

.

$H(N^j)$: the target is present with path $\#N^j$

$H(1+N^j)$: the target is absent

The data from ping k, $1 \leq k \leq j$, can be summarized in the vector $Z_k = (X_{k1}, \ldots, X_{kN})$ where X_{kn} is the single ping processor output which corresponds to the n^{th} location on ping k. The true location of the target at the time of the k^{th} ping under hypothesis H(m) will be given the symbol a(k,m).

It is desired to process the data from successive pings in a way which maximizes the probability of detection for a specified average number of false alarms. It is well known (see, for example, Reference 2) that the data processor which accomplishes this objective must compute the set of a posteriori probability density functions $p(Z_1, \ldots, Z_j | H(m))$, $m = 1, \ldots, 1+N^j$, and compare these with a threshold c which is inversely weighted by the probability that H(m) is true. That is, if

$$p(Z_1, \ldots, Z_j \,|\, H(m)) > \frac{c}{P_r\left\{H(m)\right\}}$$

then $H(m)$ is accepted whereas if

$$p(Z_1, \ldots, Z_j \mid H(m)) < \frac{C}{P_r\{H(m)\}}$$

then $H(m)$ is rejected.

We assume that the waveform $r(t)$ returned by the target consists of the transmitted narrowband signal with envelope $V(t)$, carrier frequency ω_o and random carrier phase θ plus additive, stationary, white Gaussian noise $w_1(t)$ of spectral density N_o:

$$r(t) = s(t) + w_1(t) = V(t) \cos(\omega_o t - \theta) + w_1(t)$$

The single ping processor consists of a bank of N matched filter -- envelope detector combinations, one for each location in the surveillance area. Each of the outputs has the form:

$$X = \frac{2}{N_o} \left[\left(\int_0^{T_s} r(t)\, V(t) \cos \omega_o t \, dt \right)^2 + \left(\int_0^{T_s} r(t)\, V(t) \sin \omega_o t \, dt \right)^2 \right]$$

where T_s is the duration of the transmitted signal.

If the target is not present the returned waveform consists of noise alone and the probability density function of X is given by (see Reference 6 or Section 13 of the Appendix of this report, with w_2 equal to 1):

$$p(X) = \frac{X}{\sigma^2} \exp - \frac{X^2}{2\sigma^2}$$

where $\sigma^2 = \frac{E}{2N_oW}$, E is the transmitted signal energy and W is the RF bandwidth of the receiver. When the target is present the probability density function of X is given by:

$$p(X) = \frac{X}{\sigma^2} \exp\left(-\frac{X^2+\sigma^4}{2\sigma^2}\right) I_o(X).$$

That is, X has a Rayleigh distribution with 2 degrees of freedom if the target is absent and a Rice distribution with 2 degrees of freedom if the target is present. The single ping processor outputs corresponding to different pings and/or different locations are assumed to be independent.

The optimal data processor must compute:

$$p(Z_1, \ldots, Z_j \mid H(m))$$

$$= p(X_{11}, \ldots, X_{1N}, X_{21}, \ldots, X_{2N}, \ldots, X_{j1}, \ldots, X_{jN} \mid H(m))$$

$$= \prod_{k=1}^{j} \prod_{n=1}^{N} p(X_{kn} \mid H(m)). \qquad (1)$$

Now

$$p(X_{kn} \mid H(m)) = \begin{cases} \frac{X_{kn}}{\sigma^2} \exp -\frac{X_{kn}^2}{2\sigma^2} & \text{if } n \neq a(k,m) \\ \frac{X_{kn}}{\sigma^2} \exp\left(-\frac{X_{kn}^2+\sigma^4}{2\sigma^2}\right) I_o(X_{kn}) & \text{if } n = a(k,m) \end{cases} \qquad (2)$$

That is, if n corresponds to the true target location a(k,m) on ping k under hypothesis H(m), m = 1, ..., N^j, then X is Rice distributed; otherwise it is Rayleigh distributed. Under hypothesis $1+N^j$ (target is absent), X has a Rayleigh distribution for all N locations and all j pings.

Substituting (2) in (1), we have:

$$p(Z_1, \ldots, Z_j \mid H(m))$$

$$= \prod_{k=1}^{j} \left[\prod_{\substack{n=1 \\ n \neq a(k,m)}}^{N} \frac{X_{kn}}{\sigma^2} \exp - \frac{X_{kn}^2}{2\sigma^2} \right] \left[\frac{X_{k,a(k,m)}}{\sigma^2} \exp \left(- \frac{X_{k,a(k,m)}^2 + \sigma^4}{2\sigma^2}\right) I_o(X_{k,a(k,m)} \right]$$

$$= \exp \left(- \frac{j\sigma^2}{2}\right) \prod_{k=1}^{j} \left[\prod_{n=1}^{N} \frac{X_{kn}}{\sigma^2} \exp - \frac{X_{kn}^2}{\sigma^2} \right] I_o(X_{k,a(k,m)}$$

$$m = 1, \ldots, N^j$$

and for $m = 1+N^j$:

$$p(Z_1, \ldots, Z_j \mid H(m))$$

$$= \prod_{k=1}^{j} \prod_{n=1}^{N} \frac{X_{kn}}{\sigma^2} \exp - \frac{X_{kn}^2}{2\sigma^2}$$

Those factors of $p(Z_1, \ldots, Z_j \mid H(m))$ which are the same for each hypothesis need not be computed since they will have no effect on the relative magnitudes of $p(Z_1, \ldots, Z_j \mid H(m))$ for $m = 1, \ldots, 1+N^j$. Similarly, the constant factor $\exp - \frac{j\sigma^2}{2}$ may be included in the comparison circuitry.

Thus, only the factor:

$$\prod_{k=1}^{j} I_o(X_{k,a(k,m)})$$

must be computed by the data processor and compared with a threshold for $m = 1, \ldots, N^j$:

$$\text{Accept } H(m) \text{ if: } \prod_{k=1}^{j} I_o(X_{k,a(k,m)}) > \frac{C}{P_r\{H(m)\}}$$

$$\text{and reject } H(m) \text{ if: } \prod_{k=1}^{j} I_o(X_{k,a(k,m)}) < \frac{C}{P_r\{H(m)\}}$$

If the test function does not exceed the threshold for any of the hypotheses $H(m)$, $m = 1, \ldots, N^j$, then the target is declared to be absent; that is, $H(1+N^j)$ is accepted. Decisions based upon this test function will have the desired optimal properties.

Now that the optimal test function has been obtained, any monotonic function of it may be used just as well. For example,

$$\ln \prod_{k=1}^{j} I_o(X_{k,a(k,m)}) = \sum_{k=1}^{j} \ln I_o(X_{k,a(k,m)})$$

The decision criterion becomes:

$$\text{Accept } H(m) \text{ if: } \Gamma_m(j) = \sum_{k=1}^{j} \ln I_o(X_{k,a(k,m)}) + \ln P_r\{H(m)\} > \ln C$$

(3)

$$\text{and reject } H(m) \text{ if: } \Gamma_m(j) = \sum_{k=1}^{j} \ln I_o(X_{k,a(k,m)}) + \ln P_r\{H(m)\} < \ln C$$

The path which the target follows is selected by an intelligent controller according to a plan which is unknown to the sonar operator. That is, increments in the target's path are deterministic and unknown rather than random. The maximum amount that the target location can change between pings is determined by the velocity and acceleration limitations of the target. In the absence of further information the target location on the next ping has a chance of being in any of the positions neighboring its location on the last ping. The probability $P_r\left\{a(k,m) \,\middle|\, a(k-1, m)\right\}$ that the target will be in location $a(k,m)$ on ping k when it was in location $a(k-1, m)$ on ping k-1 will in general be a function of both k and the hypothesis $H(m)$.

We assume that the target performs a random walk in the sonar surveillance area. That is, increments in the target's path from ping to ping are independent, identically distributed, zero-mean, vector valued random variables. This may be considered a worst case situation as regards knowledge of target behavior. Then:

$$
\begin{aligned}
P_r\left\{H(m)\right\} &= P_r\left\{a(1,m),\, a(2,m),\, \ldots,\, a(j,m)\right\} \\
&= P_r\left\{a(j,m) \,\middle|\, a(1,m),\, a(2,m),\, \ldots,\, a(j-1,m)\right\} \\
&\cdot\; P_r\left\{a(j-1,m) \,\middle|\, a(1,m),\, \ldots,\, a(j-2,m)\right\} \;\ldots\; P_r\left\{a(1,m)\right\} \\
&= P_r\left\{a(j,m) \,\middle|\, a(j-1,m)\right\} P_r\left\{a(j-1,m) \,\middle|\, a(j-2,m)\right\} \\
&\qquad \ldots\; P_r\left\{a(1,m)\right\} \\
&= \prod_{k=1}^{j} P_r\left\{a(k,m) \,\middle|\, a(k-1,m)\right\}
\end{aligned}
$$

Substituting into (3), the test function becomes:

$$\Gamma_m(j) = \sum_{k=1}^{j} \left[\ln I_o(X_{k,a(k,m)}) + \ln P_r \left\{ a(k,m) \;\middle|\; a(k-1,m) \right\} \right] \tag{4}$$

If the velocity of the target is constrained to some finite value, say v_m, the target location on ping k must lie within a corresponding number H of the (discrete) positions adjacent to its location on ping k-1. The test function $\Gamma_m(j)$ need not be computed for hypotheses which are inconsistent with the velocity and acceleration limitations of the target. That is, $P_r \left\{ H(m) \right\} = 0$ for certain hypotheses under the random walk assumption on target behavior. In view of the arbitrarily large and negative term resulting from $\ln P_r \left\{ H(m) \right\}$, $\Gamma_m(j)$ for these hypotheses cannot possibly exceed the threshold for any finite values of the data. Equivalently, hypotheses which are inconsistent with the target model can be disregarded at the start. Paths for which the target lies within range of its preceding location on each ping will be called "consistent" paths.

Expressions are derived in the Appendix (Section 10) for the average number of false alarms $\bar{N}_{fa}(j)$ as a function of the number of pings j and the probability of detection $P_D(j)$ as a function of j for the conditions just considered. It is also assumed for these derivations that the target remains in the sonar surveillance area for all j pings and that all consistent paths are equally likely.

The terms false alarm and detection as used in discussions of system performance in this report, merit some further explanation. A false alarm is said to occur if the test function $\Gamma_m(j)$ corresponding to target path m (hypothesis H(m)) exceeds its threshold when the target has not, in fact, occupied any of the j locations a(k,m), k = 1, ..., j, during its path through the surveillance area or is absent entirely. Thus, the probability

that H(m) will be erroneously accepted is equal to the probability that the sum of j random variables with Rayleigh distribution exceeds a threshold. A detection is said to occur if the sum of the j Rice distributed random variables corresponding to the true target path exceeds its threshold. It is clear that there are paths which agree with the true target location on certain pings and not on others. It is shown in the Appendix (Section 11) that the number of these mixed hypotheses which are consistent will be small in comparison to the total number of consistent hypotheses and they will be neglected in the derivations of expressions for system performance. Only the expressions (16), (17), (24), (25) and (26) for system performance are affected by this approximation. A modified definition of detection is discussed in Section 8 of this report which takes these mixed hypotheses into account.

3. Use of Doppler Information

The data processor can be modified to make use of doppler information when it is available. We first assume that there is no error in the doppler measurement. In this case Z_k is redefined as follows:

$$Z_k = (X_{k1}, D_{k1}, \ldots, X_{kN}, D_{kN})$$

where D_{kn} is the (exact) value of doppler associated with the n^{th} location on ping k. For a target whose acceleration and turning rate are constrained, the values of doppler on successive pings will be dependent. For simplicity, this dependence will be assumed to extend back only to the last ping. Since the noise is white, the distribution of measured doppler from non-target locations will be uniform over the RF bandwidth W of the receiver. That is,

$$p(D_{kn} \mid D_{1n}, \ldots, D_{k-1,\, n})$$

$$= \begin{cases} \frac{1}{W} \text{ rect } \left(\frac{D_{kn}}{W}\right) & \text{if} \quad n \neq a(k,m) \\ p(D_{kn} \mid D_{k-1,n}) & \text{if} \quad n = a(k,m) \end{cases}$$

where $\quad \text{rect } (t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}$

is Woodward's rect function (Reference 2). The values of doppler corresponding to different locations on the same ping are assumed to be independent of each other and of the amplitudes X_{kn}, $n = 1, \ldots, N$; $k = 1, \ldots, j$. Thus, the joint probability density function of the jN values of doppler from N locations and j pings is given by:

$$p(D_{11}, \ldots, D_{1N}, \ldots, D_{j1}, \ldots, D_{jN})$$

$$= \prod_{n=1}^{N} p(D_{1n}, \ldots, D_{jn}) = \prod_{n=1}^{N} \prod_{k=1}^{j} p(D_{kn} \mid D_{k-1,n})$$

$$= \prod_{k=1}^{j} \left[\prod_{\substack{n=1 \\ n \neq a(k,m)}}^{N} \frac{1}{W} \text{ rect } \left(\frac{D_{kn}}{W}\right) \right] p(D_{k,a(k,m)} \mid D_{k-1,\, a(k-1,m)})$$

$$= \left(\frac{1}{W}\right)^{(N-1)j} \prod_{k=1}^{j} p(D_{k,a(k,m)} \mid D_{k-1,\, a(k-1,m)}) \qquad (5)$$

The last relation in Equation (5) follows from the fact that all measured values of D_{kn} will lie within the support of rect $\frac{D_{kn}}{W}$.

Then: $\quad p(Z_1, \ldots, Z_j \mid H(m))$

$$= p(X_{11}, D_{11}, \ldots, X_{1N}, D_{1N}, \ldots, X_{j1}, D_{j1}, \ldots, X_{jN}, D_{jN} \mid H(m))$$

$$= p(X_{11}, \ldots, X_{1N}, \ldots, X_{j1}, \ldots, X_{jN} \mid H(m)) \quad p(D_{11}, \ldots, D_{1N}, \ldots, D_{j1}, \ldots D_{jN} \mid H(m))$$

$$= \exp\left(-\frac{j\sigma^2}{2}\right) \left[\prod_{k=1}^{j} \prod_{n=1}^{N} \frac{X_{kn}}{\sigma^2} \exp - \frac{X_{kn}^2}{2\sigma^2} \right] \left[\prod_{k=1}^{j} I_o(X_{k,a(k,m)}) \right]$$

$$\cdot \left[\left(\frac{1}{W}\right)^{(N-1)j} \prod_{k=1}^{j} p(D_{k,a(k,m)} \mid D_{k-1,\, a(k-1,m)}) \right] \qquad (6)$$

Once again, those factors of $p(Z_1, \ldots, Z_j \mid H(m))$ which are the same for each m can be cancelled for purposes of comparison. The test function then becomes:

$$\prod_{k=1}^{j} I_o(X_{k,a(k,m)}) \; p(D_{k,a(k,m)} \mid D_{k-1,\, a(k-1,m)})$$

which will be written:

$$\prod_{k=1}^{j} I_o(X_k) \; p(D_k \mid D_{k-1})$$

for simplicity.

The test function is to be compared with a threshold inversely weighted by the prior probabilities $P_r \left\{ H(m) \right\}$:

$$\prod_{k-1}^{j} I_o(X_k)\; p(D_k \mid D_{k-1}) \gtrless \frac{C}{P_r \left\{ H(m) \right\}}$$

or: $$\Gamma_m(j) = \sum_{k=1}^{j} \Big[\ell n\, I_o(X_k) + \ell n\, p(D_k \mid D_{k-1})$$

$$+ \ell n\, P_r \left\{ a(k,m) \mid a(k-1, m) \right\} \Big] \gtrless \ell n\, C \qquad (7)$$

The test function now contains the terms:

$$\ell n\, p(D_k \mid D_{k-1}) = p(D_{k,a(k,m)} \mid D_{k-1,\; a(k-1,m)}),$$

$$k = 1, \ldots, j$$

which weight most heavily those returns which have doppler values consistent with the last observed doppler and with the path being tested. The location factor:

$$\ell n\, P_r \left\{ a(k,m) \mid a(k-1, m) \right\}$$

is large if location $a(k,m)$ is easily accessible from location $a(k-1, m)$ in view of any available information concerning target behavior and is not dependent on the data. However, system performance can be improved by updating these location factors with the incoming doppler information. That is, $P_r \left\{ a(k, m) \mid a(k-1, m) \right\}$ is replaced by:

$$P_r \left\{ a(k,m) \,\middle|\, a(k-1,m),\ D_1,\ \ldots,\ D_{k-1} \right\}$$

$$\approx P_r \left\{ a(k,m) \,\middle|\, a(k-1,m),\ D_{k-1} \right\}$$

For the targets under consideration, this has the effect of reducing the number of consistent paths, thereby reducing the average number of false alarms. An example will illustrate the operation of the processor with uniform location factors which are updated with exact doppler information.

Suppose that the target is traveling with velocity v along a straight line toward the sonar. The maximum velocity and acceleration of the target are assumed known and equal to v_m and a_m, respectively. If a(k-1, m) was the distance between sonar and target on the last ping and T is the time between pings, then, assuming that:

$$a_m T \leq v_m - v,$$

the distance a(k, m) between sonar and target on the next ping must satisfy:

$$a(k-1,\ m) - vT - \tfrac{1}{2}\, a_m T^2 \leq a(k,\ m) \leq a(k-1,\ m) - vT + \tfrac{1}{2}\, a_m T^2$$

That is, a(k, m) is constrained to lie within an interval of length $a_m . T^2$ when the velocity (doppler) of the target is known exactly. Without doppler information a(k, m) need only satisfy:

$$a(k-1,\ m) - v_m T \leq a(k,\ m) \leq a(k-1,\ m) + v_m T$$

or a(k, m) is constrained to lie within an interval of length 2 v_m T. The ratio of the number H_d of consistent paths with doppler information to the number H without doppler information is thus:

$$\frac{H_d}{H} = \frac{a_m T}{2 v_m} \leq 1$$

Replacing H by H_d in Equation (16) results in a lower value of $\overline{N}_{fa}$. The probability of detection is thus higher when doppler information is available for a given value of $\overline{N}_{fa}$ under these conditions. More complicated target behavior including target rotation can be treated similarly.

4. Doppler Error

Let the measured doppler now consist of the true doppler D_{kn} plus the measurement error w_{kn}:

$$\hat{D}_{kn} = D_{kn} + w_{kn}$$

The jN quantities w_{kn}, $k = 1, \ldots, j$; $n = 1, \ldots, N$ are assumed to be independent, identically distributed random variables which are also independent of the true doppler D_{kn}. The data vector Z_k is now given by:

$$Z_k = (X_{k1}, \hat{D}_{k1}, \ldots, X_{kN}, \hat{D}_{kN})$$

Now:

$$p(\hat{D}_{11}, \ldots, \hat{D}_{jN} \,|\, H(m))$$

$$= \left[\prod_{k=1}^{j} \prod_{\substack{n=1 \\ n \neq a(k,m)}}^{N} \frac{1}{W} \operatorname{rect} \frac{\hat{D}_{kn}}{W} \right] \left[\prod_{k=1}^{j} p(\hat{D}_{k,a(k,m)} \,|\, \hat{D}_1, a(1,m), \ldots, \hat{D}_{k-1}, a(k-1,m)) \right]$$

$$= \frac{1}{W^{j(N-1)}} \prod_{k=1}^{j} p(\hat{D}_{k,a(k,m)} \,|\, \hat{D}_{1,\ a(1,m)}, \ldots, \hat{D}_{k-1,\ a(k-1,m)}) \qquad (8)$$

which will be written

$$\frac{1}{W^{j(N-1)}} \prod_{k=1}^{j} p(\hat{D}_k \,|\, \hat{D}_1, \ldots, \hat{D}_{k-1})$$

for simplicity. Conditioning on D_{k-1} we have:

$$p(\hat{D}_k \,|\, \hat{D}_1, \ldots, \hat{D}_{k-1})$$

$$= \int_W p(\hat{D}_k \,|\, \hat{D}_1, \ldots, \hat{D}_{k-1}, D_{k-1})\, p(D_{k-1} \,|\, \hat{D}_1, \ldots, \hat{D}_{k-1})\, d\, D_{k-1} \qquad (9)$$

where the integration is over W, the support of $p(D_{k-1})$. Then:

$$p(\hat{D}_k \,|\, \hat{D}_1, \ldots, \hat{D}_{k-1}, D_{k-1}) \approx p(\hat{D}_k \,|\, D_{k-1})$$

$$= p(D_k + w_k \,|\, D_{k-1}) = p(D_k \,|\, D_{k-1}) * p(w_k \,|\, D_{k-1})$$

$$= p(D_k \,|\, D_{k-1}) * p(w_k) \qquad (10)$$

since w_k is independent of both D_k and D_{k-1}. Substituting (10) in (9):

$$p(\hat{D}_k \,|\, \hat{D}_1, \ldots, \hat{D}_{k-1})$$

$$= \int_W \left[p(D_k \,|\, D_{k-1}) * p(w_k) \right] p(D_{k-1} \,|\, \hat{D}_1, \ldots, \hat{D}_{k-1})\, dD_{k-1}$$

Approximating $p(D_{k-1} \mid \hat{D}_1, \ldots, \hat{D}_{k-1})$ by $p(D_{k-1} \mid \hat{D}_{k-1})$ this becomes:

$$p(\hat{D}_k \mid \hat{D}_1, \ldots, \hat{D}_{k-1})$$

$$= \int_W \left[p(D_k \mid D_{k-1}) * p(w_k) \right] p(D_{k-1} \mid \hat{D}_{k-1}) \, d\, D_{k-1}$$

which upon substitution in (8) yields:

$$p(\hat{D}_{11}, \ldots, \hat{D}_{jN} \mid H(m))$$

$$= \frac{1}{W^{j(N-1)}} \prod_{k=1}^{j} \int_W \left[p(D_k \mid D_{k-1}) * p(w_k) \right] p(D_{k-1} \mid \hat{D}_{k-1}) \, d\, D_{k-1}$$

Replacing $p(D_{11}, \ldots, D_{nN} \mid H(m))$ by $p(\hat{D}_{11}, \ldots, \hat{D}_{nN} \mid H(m))$ in (6) and canceling common factors yields the test function which is to be compared with a threshold:

$$\prod_{k=1}^{j} I_o(X_k) \int_W \left[p(D_k \mid D_{k-1}) * p(w_k) \right] p(D_{k-1} \mid \hat{D}_{k-1}) \, d\, D_{k-1} \gtrless \frac{C}{P_r \{H(m)\}}$$

or:

$$\Gamma_m(j) = \sum_{k=1}^{j} \ell n \, I_o(X_k)$$

$$+ \ell n \int_W \left[p(D_k \mid D_{k-1}) * p(w_k) \right] p(D_{k-1} \mid \hat{D}_{k-1}) \, d\, D_{k-1}$$

$$+ \ell n \, P_r \left\{ a(k,m) \mid a(k-1, m) \right\} \gtrless \ell n \, C \qquad (11)$$

The random variable $\hat{D}_k$ whose conditional probability density function is given by the convolution:

$$p(D_k \mid D_{k-1}) * p(w_k) = p(D_k + w_k \mid D_{k-1}) = p(\hat{D}_k \mid D_{k-1})$$

has variance equal to:

$$\mathrm{Var}\,(\hat{D}_k \mid D_{k-1}) = \mathrm{Var}\,(D_k \mid D_{k-1}) + \mathrm{Var}\,(w_k)$$

and may be approximated by a uniform distribution about D_{k-1} with this variance. The variance of a random variable Y with uniform probability density function p(Y) is given by:

$$\mathrm{Var}\,(Y) = \frac{L^2}{12} \tag{12}$$

where L is the support of p(Y). The support L_d of the uniform probability density function $p(\hat{D}_k \mid D_{k-1})$ is thus given by:

$$L_d = \left\{ 12 \left[\mathrm{Var}\,(D_k \mid D_{k-1}) + \mathrm{Var}\,(w_k) \right] \right\}^{\frac{1}{2}}$$

so that:

$$p(\hat{D}_k \mid D_{k-1}) \approx \frac{1}{L_d} \;\mathrm{rect}\; \frac{\hat{D}_k - D_{k-1}}{L_d}$$

If the measurement error distribution $p(w_k)$ is assumed to be uniform then it follows that:

$$p(D_{k-1} \mid \hat{D}_{k-1}) = \frac{1}{L_w} \;\mathrm{rect}\; \frac{D_{k-1} - \hat{D}_{k-1}}{L_w}$$

where L_w is the support of the probability density function $p(w_k)$.

Then:

$$\int_W \left[p(D_k \mid D_{k-1}) * p(w_k) \right] \; p(D_{k-1} \mid \hat{D}_{k-1}) \, d\, D_{k-1}$$

$$= \int_W \left(\frac{1}{L_d} \operatorname{rect} \frac{\hat{D}_k - D_{k-1}}{L_d} \right) \left(\frac{1}{L_w} \operatorname{rect} \frac{D_{k-1} - \hat{D}_{k-1}}{L_w} \right) d\, D_{k-1}$$

$$= \frac{1}{L_d} \int_{\hat{D}_k - \frac{L_d}{2}}^{\hat{D}_k + \frac{L_d}{2}} \frac{1}{L_w} \operatorname{rect} \frac{D_{k-1} - \hat{D}_{k-1}}{L_w} \, d\, D_{k-1} = \frac{\Lambda}{L_d\, L_w}$$

where Λ is equal to the length of the intersection of the intervals:

$$\left[\hat{D}_k - \frac{L_d}{2} \, , \, \hat{D}_k + \frac{L_d}{2} \right] \quad \text{and} \quad \left[\hat{D}_{k-1} - \frac{L_w}{2}, \, D_{k-1} + \frac{L_w}{2} \right]$$

so that:

$$\Lambda = \begin{cases} \min \left(\hat{D}_k - \hat{D}_{k-1} + \dfrac{L_d + L_w}{2} \, , \, L_w \right) & \text{if} \quad \hat{D}_{k-1} > \hat{D}_k \\[2ex] \min \left(\hat{D}_{k-1} - \hat{D}_k + \dfrac{L_d + L_w}{2}, \, L_w \right) & \text{if} \quad \hat{D}_{k-1} < \hat{D}_k \end{cases}$$

$$= \min \left(\frac{L_d + L_w}{2} - \left| \hat{D}_k - \hat{D}_{k-1} \right| \, , \, L_w \right)$$

Substituting in (11) and dropping the common term $\ell n\, L_d\, L_w$, the test function becomes:

$$\Gamma_m(j) = \sum_{k=1}^{j} \ell n\, I_o(X_k) + \ell n \left\{ \min \left(\frac{L_d + L_w}{2} - \left| D_k - D_{k-1} \right| , L_w \right) \right\}$$

$$+ \ell n\, P_r \left\{ a(k,m) \,\middle|\, a(k-1, m) \right\}$$

where, from Equation (12):

$$\frac{L_d + L_w}{2} = \left\{ 3 \text{ Var } (w_k) \right\}^{\frac{1}{2}}$$

$$+ \left\{ 3 \left[\text{Var } (D_k \,|\, D_{k-1}) + \text{Var } (w_k) \right] \right\}^{\frac{1}{2}}$$

The second term of $\Gamma_m(j)$ is undefined for values of $\left| \hat{D}_k - \hat{D}_{k-1} \right|$ greater than $\frac{1}{2}(L_d + L_w)$. This corresponds to the fact that paths for which the measured doppler from ping to ping varies by an amount greater than $\frac{1}{2}(L_d + L_w)$ are inconsistent with assumed target limitations and doppler measurement accuracy.

5. Transient Target

Up to this point it has been assumed that the target remains in the sonar surveillance area for the duration of the pulse train. The data processor described by Equations (4) or (7) is optimum for this case; it will produce the greatest probability of detection for j pings and a specified average number of false alarms $\overline{N}_{fa}(j)$.

In a realistic situation j will not be fixed beforehand. The sonar will rather transmit pings at a constant rate for an indefinite period. This means that the threshold $C = C(j)$ must increase with the number of pings j in order to maintain the specified average number of false alarms. A

transient target which enters the surveillance area after a large number of pings has been transmitted will thus have little chance of being detected. In order to optimally process the echos returned from a transient target it is necessary to restrict the attention of the data processor to the last several, say M, pings. Assuming that a probability mass function $p(\delta)$ is known for the length of time δ that the target spends in the surveillance area, M can be chosen to maximize the probability of detection $P(M)$ of the data processor for a "window" of length M pings.

Conditioning on we have:

$$P(M) = \sum_{\delta=0}^{\infty} P(M \mid \delta) \; p(\delta) \tag{13}$$

where $P(M \mid \delta)$ is the probability of detection given that the target remains in the surveillance area for δ pings. Now:

$$P(M \mid \delta) = \begin{cases} P_r \left\{ \Gamma(\delta) > C(M) \right\} & \text{for } \delta < M \\ P_r \left\{ \Gamma(M) > C(M) \right\} & \text{for } \delta > M \end{cases}$$

Substituting in (13) we have:

$$P(M) = P_r \left\{ \Gamma(M) > C(M) \right\} \; P_r \left\{ \delta > M \right\} + \sum_{\delta=0}^{M} P_r \left\{ \Gamma(\delta) > C(M) \right\} \; p(\delta)$$

The threshold $C(M)$ is set to establish a tolerable false alarm rate and is implicitly defined through a rather complicated expression (16). The optimal window size M thus cannot be determined analytically but must be obtained by plotting the above expression for $P(M)$ from curves of $C(j)$ vs. j and

$P_r \left\{ \Gamma(j) > C \right\}$ vs. C and choosing the M for which $P(M)$ is maximized. The chief difficulty in this approach is the unavailability of $p(\delta)$. It should be recalled, too, that data storage requirements increase exponentially with M and may dictate that a suboptimal value of M be used.

6. Fading Target

The additive Gaussian noise channel which has served as a model so far in this report does not adequately represent a situation in which the target aspect may vary from ping to ping. The returned signal level from a grossly non-spherical target will build up and decrease as the target presents now a broadside, now a head-on aspect to the sonar. The performance of the sonar system will be noticeably poorer under these circumstances. A simple model for a target with uniformly varying aspect is a (two-dimensional) rectangle which is orthogonal to the horizontal plane and whose angle with the line between sonar and target is a random variable ψ with probability density function $p(\psi)$ (see Figure 1).

Figure 1

UVTA (FADING) TARGET MODEL

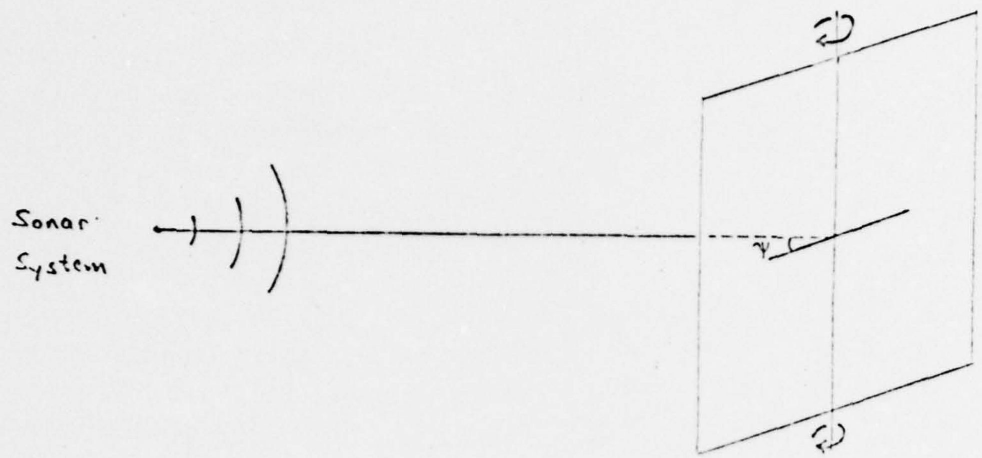


FIGURE 1

The returned waveform $\tilde{r}(t)$ under uniformly varying target aspect (UVTA) conditions is then given by:

$$\tilde{r}(t) = (\sin \psi(t))\ s(t) + w_1(t) = w_2(t)\ s(t) + w_1(t)$$

The channel model is seen to now include the multiplicative noise $w_2(t)$ as well as the additive noise $w_1(t)$.

The random variable ψ is assumed to be chosen from a uniform distribution over the interval $[0, \pi]$. Values of ψ are independent from ping to ping and remain constant for the duration of the sonar pulse. The random process $\psi(t) = \psi(j)$ is thus continuous valued with discrete parameter j. The probability density function of $w_2 = \sin \psi$ is shown in Section 12 of the Appendix to be:

$$p(w_2) = \frac{2}{\pi} (1 - w_2^{\,2})^{-\frac{1}{2}}, \qquad 0 \leq w_2 \leq 1.$$

The resulting single ping processor output $\tilde{X}$ with target absent has a Rayleigh distribution as before, but when the target is present, $\tilde{X}$ now has a probability density function given by (see Appendix, Section 13):

$$p(\tilde{X}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta(m,n)\ \tilde{X}^{2n+1} \exp - \frac{\tilde{X}^2}{2\sigma^2} \qquad (14)$$

where:

$$\beta(m,n) = \frac{(-1)^m (\sigma^2)^{m-1} (1;\ 2;\ m+n)}{m!\,(n!)^2\,(m+n)!\ \ 2^{2m+3n}}$$

and:
$$(1;\ 2;\ m+n) = \begin{cases} 1 & \text{if} \quad m+n = 0 \\ \prod_{r=1}^{m+n} 1 + 2(r-1) & \text{if} \quad m+n \neq 0 \end{cases}$$

Ignoring doppler information for the moment, the optimal processor must compute:

$$p(Z_1, \ldots, Z_j \mid H(m)) = \prod_{k=1}^{j} \prod_{n=1}^{N} p(\tilde{X}_{kn} \mid H(m))$$

$$= \prod_{k=1}^{j} \left[\prod_{\substack{n=1 \\ n \neq a(k,m)}}^{N} \frac{\tilde{X}_{kn}}{\sigma^2} \exp - \frac{\tilde{X}_{kn}^2}{2\sigma^2} \right] p(\tilde{X}_{k,a(k,m)})$$

$$= \prod_{k=1}^{j} \left[\prod_{\substack{n=1 \\ n \neq a(k,m)}}^{N} \frac{\tilde{X}_{kn}}{\sigma^2} \exp - \frac{\tilde{X}_{kn}^2}{2\sigma^2} \right] \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta(m,n)\, \tilde{X}_{k,a(k,m)}^{2n+1} \exp - \frac{\tilde{X}_{k,a(k,m)}^2}{2\sigma^2}$$

$$= \prod_{k=1}^{j} \left[\prod_{n=1}^{N} \frac{\tilde{X}_{kn}}{\sigma^2} \exp - \frac{\tilde{X}_{kn}^2}{2\sigma^2} \right] \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sigma^2 \beta(m,n)\, \tilde{X}_{k,a(k,m)}^{2n}$$

Canceling factors common to all hypotheses we obtain:

$$\prod_{k=1}^{j} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sigma^2 \beta(m,n)\, \tilde{X}_{k,a(k,m)}^{2n}$$

which will be written:

$$\prod_{k=1}^{j} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sigma^2 \beta(m,n) \tilde{X}_k^{2n}$$

for simplicity.

Taking the logarithm of this quantity results in the following test function:

$$\Gamma(j) = \sum_{k=1}^{j} \ln \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sigma^2 \beta(m,n) \tilde{X}_k^{2n}$$

$$= \sum_{k=1}^{j} \ln \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m (\sigma^2)^m (1;\ 2;\ m+n)}{m!\,(n!)^2\,(m+n)!\ 2^{2m+3n}} \tilde{X}_k^{2n}$$

$$= \sum_{k=1}^{j} f(\tilde{X}_k)$$

which is seen to vary with $\sigma^2 = \frac{E}{2N_oW}$.

The weighting function $f(\tilde{X}_k)$ for UVTA conditions includes a constant term which is obtained by setting $\tilde{X}_k = 0$:

$$f(0) = \ln \sum_{m=0}^{\infty} \frac{(-1)^m (\sigma^2)^m (1;\ 2;\ m)}{(m!)^2\ 2^{2m}} \neq 0$$

This constant can be included in the comparison circuitry; that is:

$$\Gamma'(j) = \sum_{k=1}^{j} f(\tilde{X}_k) \gtrless C(j)$$

is equivalent to:

$$\sum_{k=1}^{j} \left[f(\tilde{X}_k) - f(0) \right] \gtrless C - j\,f(0) = C'(j)$$

Curves of $f(X) - f(0)$ vs. X are plotted in Figure (2) with σ^2 as a parameter. The flat lower region of each of these curves, for which $f(X) - f(0)$ has negligibly small values, may be approximated by the horizontal axis as is shown in Figure (3) for $\sigma^2 = 4$. That is, the optimal processor for UVTA conditions has the effect of thresholding (assigning a value of zero to) weak returns from the fading target. Figure (2) indicates that the threshold c should increase with increasing additive signal to noise ratio σ^2.

Figure 2

Figure 3

OPTIMAL AMPLITUDE WEIGHTING FUNCTION

UNDER FADING CONDITIONS

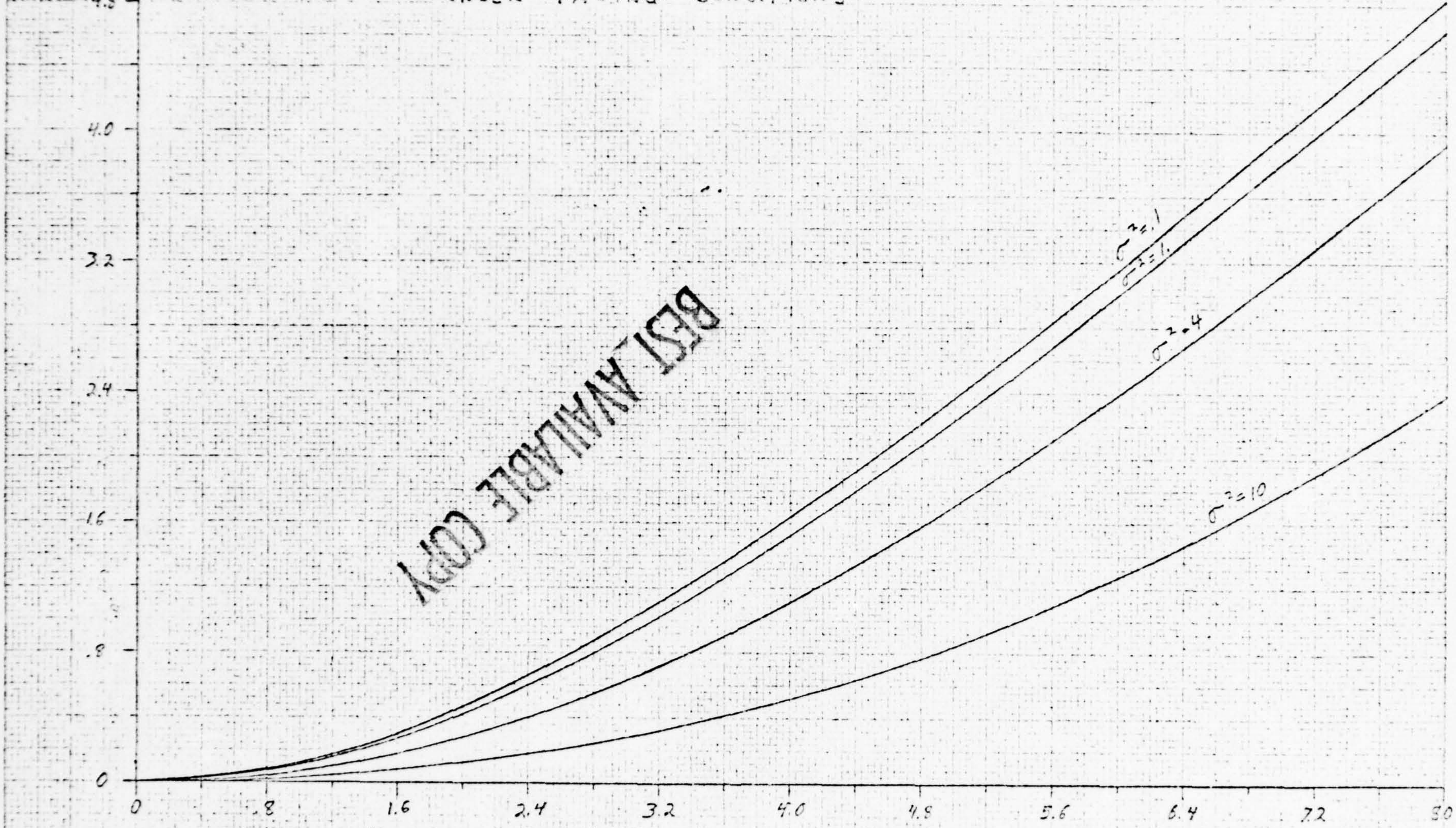
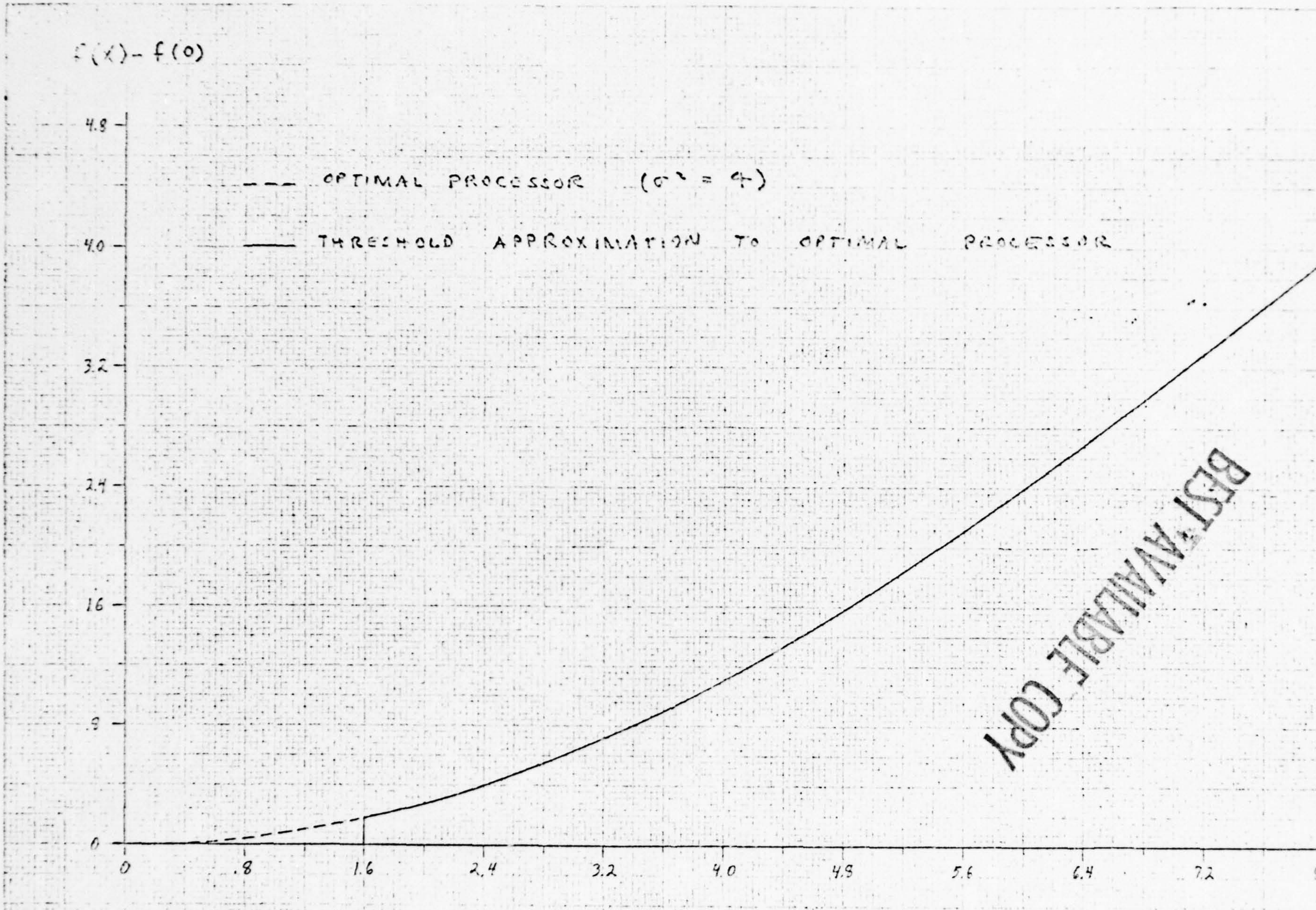


FIGURE 2



f(x) - f(0)
4.8
4.0
3.2
2.4
1.6
.8
0
0
.8
1.6
2.4
3.2
4.0
4.8
5.6
6.4
7.2
8.0
X
--- OPTIMAL PROCESSOR ($\sigma^2 = 4$)
— THRESHOLD APPROXIMATION TO OPTIMAL PROCESSOR

FIGURE 3

One of the detection schemes which has been proposed for use with a fading target is the "r-out-of-j" processor (see Reference 3). In this scheme the target is declared to be present with path m (hypothesis H(m) is accepted) if r of the single ping processor outputs corresponding to the j locations of path m exceed a threshold c. This operation can be written formally as follows:

Accept H(m) if: $$\sum_{k=1}^{j} u(\tilde{X}_k - c) > r$$

and reject H(m) if: $$\sum_{k-1}^{j} u(\tilde{X}_k - c) < r$$

where

$$u(X) = \begin{cases} 1 & \text{if} \quad X \geq 0 \\ 0 & \text{if} \quad X < 0 \end{cases}$$

The weighting function:

$$u\,(\tilde{X}_k - c)$$

corresponding to this scheme is graphed with c = 2.0 in Figure (4) together with the optimal weighting function derived above for UVTA conditions with a large value of the additive signal-to-noise ratio ($\sigma^2 = 10$).

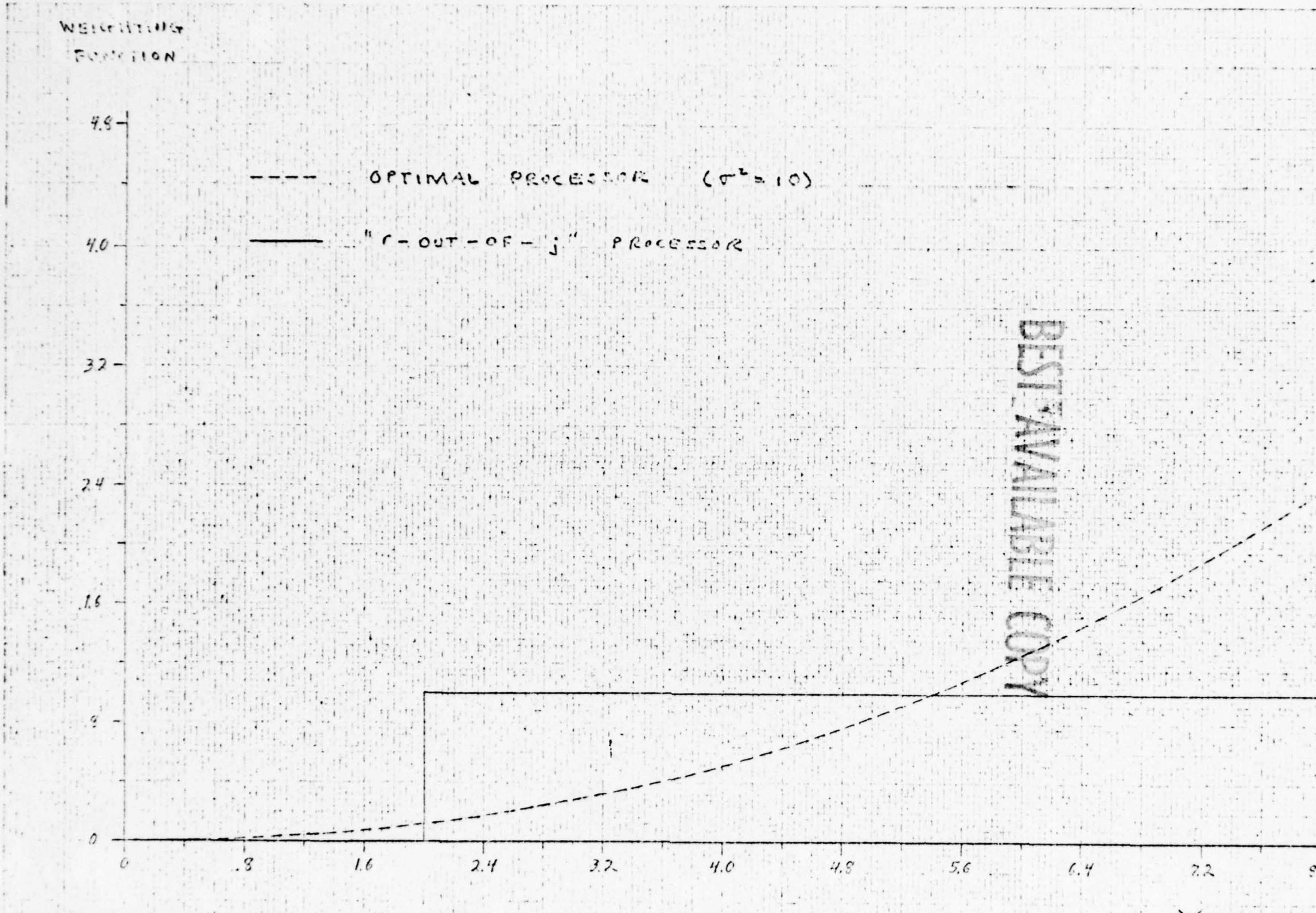

WEIGHTING FUNCTION
4.8
4.0
3.2
2.4
1.6
.8
0
OPTIMAL PROCESSOR ($\sigma^2 = 10$)
"r-OUT-OF-j" PROCESSOR
0
.8
1.6
2.4
3.2
4.0
4.8
5.6
6.4
7.2
8.0
X
BEST AVAILABLE COPY

FIGURE 4

Figure 4

Both processors are seen to give very little weight to weak returns ($X_k < c$). However, the optimal processor gives stronger returns a proportionately greater weight whereas the r-out-of-j processor weights strong returns uniformly. The two processors differ more radically for lower values of σ^2 as can be deduced from Figure (2). For j = 1 the r-out-of-j processor and the optimal processor are identical since both processors merely compare the single ping processor output X with a threshold. It has been shown in previous work (Reference 8) that the r-out-of-j processor performs best when j = 1. From Figure 7 it is seen that the optimal processor performance is enhanced as j is increased.

Expressions for the performance of a square law processor with thresholded data are derived in Sections 14 and 15 of the Appendix for the conditions of additive noise and additive plus multiplicative noise (UVTA conditions).

7. Data Storage Considerations

There are H extensions of the NH^{j-2} consistent paths on ping j-1 which result in the NH^{j-1} consistent paths for ping j. The NH^{j-1} values of $\Gamma(j)$ for ping j can be obtained recursively from the NH^{j-2} values of $\Gamma(j-1)$ on ping j-1 by adding the appropriate term to $\Gamma(j-1)$. However, NH^{j-1} quantities must still be stored on the j^{th} ping and so the storage requirements increase exponentially with the number of pings.

Since it has been assumed that only one target lies within the surveillance area, necessarily all or all but one of the single ping processor outputs corresponding to the H consistent path extensions result from noise alone. It may be desired to evaluate the test function $\Gamma(j)$ for only μ of the H consistent path extensions and dump the data stored for the remaining H-μ consistent path extensions. This will decrease the number of stored quantities after j pings from NH^{j-1} to $N\mu^{j-1}$; that is, by a factor of $(\frac{\mu}{H})^{j-1}$. The μ path extensions are chosen according to a maximum likelihood ranking procedure. This means that the μ path extensions which have largest test function $\Gamma(j)$ are retained. Expressions for the resulting system performance as a function of H and μ are given in Section 16 of the Appendix for the special case of uniform doppler and location factors and for both the additive and UVTA channel models.

8. Modified Definition of Detection

A "detection" has been defined to be the event that the integrated returns from the true target path exceed a threshold. Other definitions of a detection are reasonable. For example, if the threshold is crossed by the returns from a consistent path which agrees with the true target path at j-k points and differs from the true target path at k points at

which it is within a specified distance of the true target location, then this may also be called a detection. That is, the hypothesis (path) which has been accepted differs only slightly from the true hypothesis (true target path). More generally, a detection may be defined to be the event of a threshold crossing by the test function corresponding to any element of a subset Ω of the original NH^{j-1} consistent hypotheses. In the example above, Ω is defined as follows: Hypothesis H(m) is contained in Ω if each point of H(m) is within ϵ of the true target location. The quantity ϵ must be small enough to insure that all elements of Ω are consistent. Ω has h^j elements under this definition where h is the number of (discrete) locations within radius ϵ of the true target location.

9. Conclusion

The form of the theoretically optimum data processor for multiple ping sonar has been derived under certain stated assumptions about the channel and target. Suboptimal schemes have been discussed and the resulting performance has been analyzed and presented in the Appendix of this report.

APPENDIX

10. Performance of Optimal Processor in Additive Noise

Since all paths are equally likely, the optimal processor must only compute:

$$\Gamma(j) = \sum_{k=1}^{j} \ln I_o(X)$$

for all consistent hypotheses. For $X << 1$ the following approximation is valid:

$$\ln I_o(X) \approx \frac{1}{4} X^2$$

That is, the optimal processor can be replaced for small X by a square law detector (the constant factor $\frac{1}{4}$ can be dropped since it appears in the test function of each hypothesis). This approximation is not valid for large values of X but will be used to simplify the analysis. It may be argued that the degradation incurred by this approximation is less important when X is large and detection is already assured. Thus:

$$\Gamma(j) = \sum_{k=1}^{j} \ln I_o(X) \approx \frac{1}{4} \sum_{k=1}^{j} X^2$$

If noise alone is present X has a Rayleigh distribution and the characteristic function of X^2 is (Reference 4):

$$\Phi(u) = \frac{1}{1 - 2iu\sigma^2}$$

Since X is independent from ping to ping, X^2 is also independent from ping to ping and the characteristic function $\Phi_{\Gamma}(\cdot)$ of $\Gamma(j)$ is given by:

$$\Phi_{\Gamma}(u) = \left[\Phi(u)\right]^j = \left[\frac{1}{1 - 2iu\sigma^2}\right]^j$$

This can be inverted by using Pair 431 of Reference (5) to obtain the probability density function $p_o(\cdot)$ of $\Gamma(j)$ when noise alone is present:

$$p_o(\gamma) = \frac{1}{(2\sigma^2)^j\,(j-1)!}\quad \gamma^{j-1}\ \exp - \frac{\gamma}{2\sigma^2}\ ,\quad \gamma > 0$$

The probability that $\Gamma(j)$ exceeds the threshold C for a particular noise path is:

$$\int_C^{\infty} p_o(\gamma)\,d\gamma = \frac{1}{(2\sigma^2)^j\,(j-1)!}\int_C^{\infty} \gamma^{j-1}\ \exp - \frac{\gamma}{2\sigma^2}\ d\gamma$$

$$= 1 - I\left(\frac{C}{2\sigma^2\sqrt{j}}\ , j-1\right)$$

where:

$$I\left(\frac{v}{\sqrt{n}}\ , n-1\right) = \int_0^v \frac{y^{n-1}\,e^{-y}}{(n-1)!}\,d\,y$$

is Pearson's Incomplete Gamma Function Ratio.

There are NH^{j-1} consistent paths after j pings have been transmitted. Since the test functions $\Gamma_m(j)$, $m = 1, \ldots, NH^{j-1}$, corresponding to these paths are approximately independent for large H, the probability that n of these result in false alarms becomes:

$$P_r\left\{N_{fa}(j)=n\right\}$$

$$=\binom{NH^{j-1}}{n}\left[P_r\left\{\Gamma(j)>C\right\}\right]^n\left[1-P_r\left\{\Gamma(j)>C\right\}\right]^{NH^{j-1}-n}$$

The average number of false alarms is then given by:

$$\bar{N}_{fa}(j) = \sum_{n=1}^{NH^{j-1}} n\, P_r\left\{N_{fa}(j)=n\right\}$$

$$=\sum_{n=1}^{NH^{j-1}} n\binom{NH^{j-1}}{n}\left[P_r\left\{\Gamma(j)>C\right\}\right]^n\left[1-P_r\left\{\Gamma(j)>C\right\}\right]^{NH^{j-1}-n} \qquad (15)$$

$$=\sum_{n=1}^{NH^{j-1}} n\binom{NH^{j-1}}{n}\left[1-I\left(\frac{C}{2\sigma^2\sqrt{j}},\ j-1\right)\right]^n\left[I\left(\frac{C}{2\sigma^2\sqrt{j}},\ j-1\right)\right]^{NH^{j-1}-n} \qquad (16)$$

For the true target path X has a Rice distribution with mean and variance both equal to σ^2 (Reference 6) and the characteristic function of X^2 is (Reference 4):

$$\Phi(u) = \frac{1}{1-2iu\sigma^2}\exp\frac{iu\sigma^4}{1-2iu\sigma^2}$$

from which we obtain the characteristic function of $\Gamma(j)$:

$$\Phi_{\Gamma}(u) = \left[\Phi(u)\right]^j$$

$$= \left[\frac{1}{1-2iu\sigma^2} \exp \frac{iu\sigma^4}{1-2iu\sigma^2}\right]^j$$

This can be inverted by using Pair 650.0 of Reference 5 to obtain the probability density function of $\Gamma(j)$ for the true target path:

$$p_1(\gamma) = \frac{\exp - j\sigma^2/2}{(2\sigma^2)^j} \left(\frac{4\gamma}{j}\right)^{\frac{j-1}{2}} \exp\left(-\frac{\gamma}{2\sigma^2}\right) I_{j-1}\left(\sqrt{j\gamma}\right), \quad \gamma > 0$$

where $I_j(\cdot)$ is the modified Bessel function of j^{th} order. Then:

$$P_D(j) = \int_C^{\infty} p_1(\gamma)\, d\gamma$$

$$= \int_C^{\infty} \frac{\exp(-j\sigma^2/2)}{(2\sigma^2)^j} \left(\frac{4\gamma}{j}\right)^{\frac{j-1}{2}} \exp\left(-\frac{\gamma}{2\sigma^2}\right) I_{j-1}\left(\sqrt{j\gamma}\right) d\gamma$$

$$= \left(\frac{2}{\sigma\sqrt{j}}\right)^{j-1} Q_j\left(\sigma\sqrt{j}, \frac{\sqrt{C}}{\sigma}\right) \qquad (17)$$

where:

$$Q_j(\alpha, \beta) = \int_{\beta}^{\infty} t\left(\frac{t}{2}\right)^{j-1} \exp\left(-\frac{t^2+\alpha^2}{2}\right) I_{j-1}(\alpha t)\, dt$$

is Marcum's j^{th} Q function. The probability of detection $P_D(j)$ is plotted vs. the average number of false alarms, $\bar{N}_{fa}(j)$, with j as a parameter in Figure 5.

11. Number of Mixed Hypotheses

A mixed path (hypothesis) is one which agrees with the true target path on k pings and differs from the true target path on j-k pings where $1 \leq k \leq j-1$. There are NH^{j-1} consistent paths in all and $(N-1)(H-1)^{j-1}$ of these have no points in common with the one true target path. The remaining $NH^{j-1} - (N-1)(H-1)^{j-1} - 1$ paths have one or more points in common with the true target path.

Let N_m be the number of mixed paths. Then:

$$\frac{N_m}{NH^{j-1}} = \frac{NH^{j-1} - (N-1)(H-1)^{j-1} - 1}{NH^{j-1}}$$

$$\approx \frac{NH^{j-1} - (N-1)(H-1)^{j-1}}{NH^{j-1}} = 1 - \frac{N-1}{N}\left(\frac{H-1}{H}\right)^{j-1}$$

For large H:

$$\left(\frac{H-1}{H}\right)^{j-1} \approx 1 - \frac{j-1}{H}$$

so that:

$$\frac{N_m}{NH^{j-1}} \approx 1 - \frac{N-1}{N}\left(1 - \frac{j-1}{H}\right) \approx \frac{j-1}{H} .$$

since N is large. In this case j is the number of pings being processed and is the same as the length of the window M of Section 5. Data storage requirements will limit j to a value of 10 or less and H will typically be equal to 100 or more. Thus:

$$\frac{N_m}{NH^{j-1}} < 0.1$$

12. Distribution of w_2

Let $G(\cdot)$ be the distribution function of w_2 and $F(\cdot)$ the distribution function of ψ'. The random variable ψ' is uniformly distributed on $[0,\pi]$ and $w_2 = \sin \psi'$.

$$G(w) = P_r \left\{ w_2 \leq w \right\} = P_r \left\{ \sin \psi' \leq w \right\}$$

$$= P_r \left\{ \psi \leq \sin^{-1} w \right\} + P_r \left\{ \psi \geq \pi - \sin^{-1} w \right\}$$

$$= F(\sin^{-1} w) + 1 - F(\pi - \sin^{-1} w), \quad 0 \leq w \leq 1.$$

The probability density function of w_2 is then given by:

$$p(w_2) = \frac{d}{d\,w_2} \; G(w_2)$$

$$= \frac{2}{\pi} \, (1 - w_2^2)^{-\frac{1}{2}}, \quad 0 \leq w_2 \leq 1.$$

13. Distribution of Single Ping Processor Output under UVTA Conditions

The returned single ping waveform is given by:

$$\tilde{r}(t) = w_2 \, S(t, \theta) + w_1(t) \tag{18}$$

Let $\tilde{R}$ be a vector whose components are the time sample values of $\tilde{r}(t)$. Similarly, let $S(\theta)$ and W_1 be the time sample vectors of $s(\theta,t)$ and $w_1(t)$. Let K be the covariance matrix of the time samples of the additive noise $w_1(t)$. Then (18) can be written in the form:

$$\tilde{R} = w_2 \, S(\theta) + W_1$$

Now:

$$s(t,\theta) = V(t) \cos (\omega_o t - \theta)$$

$$= \left[V(t) \cos \omega_o t\right] \cos \theta + \left[V(t) \sin \omega_o t\right] \sin \theta$$

$$= V_c(t) \cos \theta + V_s(t) \sin \theta \qquad (19)$$

Let V_c and V_s denote the vectors of time sampled values of $V_c(t)$ and $V_s(t)$, respectively. The single ping processor consists of a matched filter followed by an envelope detector. When the input to the single ping processor is $\tilde{R}$, the output $\tilde{X}$ is given by:

$$\tilde{X} = \left[(\tilde{R}' K^{-1} V_c)^2 + (\tilde{R}' K^{-1} V_s)^2\right]^{\frac{1}{2}}$$

where $(\ \ )'$ denotes transpose.

The quantities $\tilde{R}' K^{-1} V_c$ and $\tilde{R}' K^{-1} V_s$ are linear combinations of Gaussian random variables and are therefore Gaussian. From Equation (19) we see that they are 90 degrees out of phase and are thus uncorrelated. Since they are Gaussian random variables, they are therefore also independent.

When the target echo is present $\tilde{R} = w_2 S(\theta) + w_1$ and in this case, for a given value of w_2, we have:

$$E\,(\tilde{R}' K^{-1} V_c) = E(w_2\, S(\theta) + W_1)'\, K^{-1}\, V_c$$

$$= w_2\, S'(\theta)\, K^{-1}\, V_c$$

and

$$E(\tilde{R}' K^{-1} V_s) = w_2 S'(\theta) K^{-1} V_s$$

When $\theta = 0$ degrees, $S(\theta) = V_c$ and:

$$w_2 S'(\theta) K^{-1} V_c = w_2 V_c K^{-1} V_c$$

$$= w_2 \sigma^2$$

where $\sigma^2 = V_c' K^{-1} V_c = V_s' K^{-1} V_s$. Since the noise is white with spectral density N_o we have:

$$K^{-1} = \frac{1}{N_o W}$$

where W is the RF bandwidth of the receiver.

Then:

$$\sigma^2 = \frac{V_c' V_c}{N_o W} = \frac{V_s' V_s}{N_o W} = \frac{E}{2 N_o W}$$

where E is the energy of the transmitted signal.

Also:

$$w_2 S'(\theta) K^{-1} V_s = w_2 V_c' K^{-1} V_s = 0$$

since V_c is orthogonal to V_s (see Equation (19)). When $\theta = 90$ degrees, $S(\theta) = V_s$ and:

$$w_2 S'(\theta) K^{-1} V_s = w_2 V_s K^{-1} V_s = w_2 \sigma^2$$

and:

$$w_2 S'(\theta) K^{-1} V_c = w_2 V_s K^{-1} V_c = 0$$

Thus, the means are proportional to σ^2 and 90 degrees out of phase. The variance of $R K^{-1} V_c$ with target present is:

$$\mathrm{Var}\,(\tilde{R}' K^{-1} V_c) =$$

$$= E\,(\tilde{R}' K^{-1} V_c - E\,\tilde{R}' K^{-1} V_c)' \,(\tilde{R}' K^{-1} V_c - E\,\tilde{R}' K^{-1} V_c)$$

$$= E\,(W_1' K^{-1} V_c)' \,(W_1' K^{-1} V_c) = \sigma^2$$

$$= \mathrm{Var}\,(\tilde{R}' K^{-1} V_s)$$

The square root of the sum of the squares of two Gaussian random variables with equal variance and means equal to a sin ϕ and a cos ϕ, respectively, has a Rice distribution (Reference 7) with probability density function equal to:

$$p(y) = \frac{y}{\mathrm{Var}\,(y)} \exp\left(- \frac{y^2 + a^2}{2\,\mathrm{Var}(y)}\right) I_0 \left(\frac{a\,y}{\mathrm{Var}\,(y)}\right)$$

the probability density function of $\tilde{X}$ given w_2 is thus given by:

$$p\,(\tilde{X} \mid w_2) = \frac{\tilde{X}}{\sigma^2} \exp\left(- \frac{\tilde{X}^2 + (w_2\,\sigma^2)^2}{2\,\sigma^2}\right) I_0 \left(\frac{w_2\,\sigma^2\,\tilde{X}}{\sigma^2} \right)$$

$$= \frac{\tilde{X}}{\sigma^2} \exp\left(- \frac{\tilde{X}^2 + w_2^{\,2}\,\sigma^4}{2\,\sigma^2}\right) I_0 \,(w_2\,\tilde{X})$$

and so:

$$p(\tilde{X}) = \int_0^1 p(\tilde{X} \mid w_2)\ p(w_2)\ d\,w_2$$

$$= \frac{2}{\pi} \int_0^1 \frac{\tilde{X}}{\sigma^2} \exp\left(- \frac{\tilde{X}^2 + w_2^{\,2}\,\sigma^4}{2\sigma^2}\right) I_o (w_2 \hat{X}) (1 - w_2^{\,2})^{-\frac{1}{2}}\ d\,w_2$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{(-1)^m (\sigma^2)^{m-1} (1;\ 2;\ m+n)}{m!\ (n!)^2\ (m+n)!\ 2^{2m+3n}} \right\} \tilde{X}^{\,2n+1} \exp - \frac{\tilde{X}^2}{2\sigma^2}$$

where:

$$(1;\ 2;\ m+n) = \begin{cases} 1 & \text{if} \quad m+n = 0 \\ \prod_{r=1}^{m+n} 1 + 2(r-1) & \text{if} \quad m+n \neq 0 \end{cases}$$

Letting:

$$\beta(m,n) = \frac{(-1)^m (\sigma^2)^{m-1} (1;\ 2;\ m+n)}{m!\ (n!)^2\ (m+n)!\ \ 2^{2m+3n}}$$

this becomes:

$$p(\tilde{X}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta(m,n)\, \tilde{X}^{\,2n+1} \exp - \frac{\tilde{X}^2}{2\sigma^2} \qquad (20)$$

14. Performance of Threshold-Square Law Processor in Additive Noise

The single ping processor output X is thresholded at c; that is, only those outputs which exceed c are passed on for multiple ping processing. The probability density function of the thresholded random variable $\hat{X}$ is given by:

$$\hat{p}\,(\hat{X}) = P_r \left\{ X < c \right\} \, \delta\,(\hat{X}) + p\,(\hat{X})\; u\,(\hat{X} - c)$$

where $p(\cdot)$ is the probability density function of X before thresholding, $\delta(\cdot)$ is the Dirac delta function and:

$$u\,(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y \leq 0 \end{cases}$$

Let $G(\cdot)$ be the distribution function of $\hat{X}^2$. Then:

$$G(\xi) = P_r \left\{ X^2 \leq \xi \right\} = P_r \left\{ \hat{X} \leq \sqrt{\xi} \right\}$$

$$= \begin{cases} 0 & \text{if } \xi \leq 0 \\ P_r \left\{ X \leq c \right\} & \text{if } 0 < \sqrt{\xi} \leq c \\ P_r \left\{ X \leq \sqrt{\xi} \right\} & \text{if } c < \sqrt{\xi} \end{cases} \qquad (21)$$

The probability density function $q(\cdot)$ of $\hat{X}^2 = \xi$ is obtained by differentiating (21):

$$q(\xi) = \frac{d}{d\xi} G(\xi)$$

$$= P_r \left\{ X \leq c \right\} \delta(\xi) + \frac{1}{2\sqrt{\xi}} p(\sqrt{\xi}) u(\sqrt{\xi} - c)$$

$$= P_r \left\{ X \leq c \right\} \delta(\xi) + \frac{1}{2\sqrt{\xi}} p(\sqrt{\xi}) u(\xi - c^2) \qquad (22)$$

When the input to the single ping processor consists of noise alone, X has a Rayleigh distribution:

$$p(X) = \frac{X}{\sigma^2} \exp - \frac{X^2}{2\sigma^2}$$

and so:

$$q(\xi) = \left[\int_0^c \frac{X}{\sigma^2} \exp\left(- \frac{X^2}{2\sigma^2} \right) d X \right] \delta(\xi)$$

$$+ \frac{1}{2\sqrt{\xi}} \frac{\sqrt{\xi}}{\sigma^2} \left(\exp - \frac{\xi}{2\sigma^2}\right) u(\xi - c^2)$$

$$= \left(1 - \exp - \frac{c^2}{2\sigma^2}\right) \delta(\xi) + \frac{1}{2\sigma^2} \left(\exp - \frac{\xi}{2\sigma^2} \right) u(\xi - c^2)$$

The characteristic function of $\hat{X}^2$ is then:

$$\Phi(u) = E \exp(i u \hat{X}^2) = \int_0^\infty \exp(i u \xi) \, q(\xi) \, d\xi$$

$$= 1 - \exp\left(- \frac{c^2}{2\sigma^2}\right) + \frac{1}{1 - 2iu\sigma^2} \exp \left[- \frac{c^2}{2\sigma^2} (1 - 2iu\sigma^2) \right]$$

Since:

$$\Gamma(j) \approx \sum_{k=1}^{j} \hat{x}_k^{2}$$

the characteristic function of $\Gamma(j)$ is equal to:

$$\Phi_{\Gamma}(u) = \left[\Phi(u) \right]^{j}$$

$$= \sum_{k=0}^{j} \binom{j}{k} \left[1 - \exp - \frac{c^2}{2\sigma^2} \right]^{j-k} \left[\frac{1}{1 - 2iu\sigma^2} \exp\left(- \frac{c^2}{2\sigma^2} (1 - 2iu\sigma^2) \right) \right]^{k}$$

This can be inverted directly to obtain the probability density function of $\Gamma(j)$:

$$p(\gamma) = \sum_{k=0}^{j} \binom{j}{k} \left[1 - \exp - \frac{c^2}{2\sigma^2} \right]^{j-k} \frac{\exp\left(- k\, c^2/2\sigma^2\right)}{(2\sigma^2)^k\,(k-1)!}$$

$$\exp\left(\frac{k\, c^2 - \gamma}{2\sigma^2}\right) \left[\gamma - k\, c^2 \right]^{k-1}, \qquad \gamma > k\, c^2$$

Then, for a particular noise path:

$$P_r \left\{ \Gamma(j) > C \right\} = \int_{C}^{\infty} p(\gamma)\, d\gamma$$

$$= \sum_{k=1}^{j} \binom{j}{k} \left[1 - \exp - \frac{c^2}{2\sigma^2} \right]^{j-k} \exp\left(- \frac{k\, c^2}{2\sigma^2} \right)$$

$$\cdot \left[1 - I\left(\frac{\tilde{c} - k\, c^2}{2\sigma^2 \sqrt{k}}, k - 1 \right) \right] \qquad (23)$$

where:

$$\tilde{C} = \max \left\{C,\ k\ c^2\right\}$$

The average number of false alarms is given by Equation (15) and is repeated below:

$$\bar{N}_{fa}(j) = \sum_{n=1}^{NH^{j-1}} n \binom{NH^{j-1}}{n} \left[P_r \left\{\Gamma(j) > C\right\}\right]^n \left[1 - P_r \left\{\Gamma(j) > C\right\}\right]^{NH^{j-1} - n} \tag{24}$$

where now $P_r \left\{\Gamma(j) > C\right\}$ is given by Equation (23).

When the target is present X has a Rice distribution:

$$p(X) = \frac{X}{\sigma^2} \exp\left(-\frac{X^2 + \sigma^4}{2\sigma^2}\right) I_o(X)$$

and for this case, from Equation (22), the probability density function of $\hat{X}^2 = \xi$ is given by:

$$q(\xi) = \left[\int_o^c \frac{X}{\sigma^2} \exp\left(-\frac{X^2 + \sigma^4}{2\sigma^2}\right) I_o(X)\, d\, X\right] \delta(\xi)$$

$$+ \frac{1}{2\sigma^2} \exp\left(-\frac{\xi + \sigma^4}{2\sigma^2}\right) I_o(\xi)\ u(\sqrt{\xi} - c)$$

$$= \left[1 - Q\left(\sigma, \frac{c}{\sigma}\right)\right] \delta(\xi)$$

$$+ \frac{1}{2\sigma^2} \exp\left(-\frac{\xi + \sigma^4}{2\sigma^2}\right) I_o(\xi)\ u(\xi - c^2)$$

the characteristic function of $\hat{X}^2$ is then:

$$\Phi(u) = \int_0^\infty \exp(i u \xi)\ q(\xi)\, d\xi$$

$$= 1 - Q(\sigma, c/\sigma) + \exp\left[-\frac{\sigma^2}{2} - \frac{c^2}{2\sigma^2}(1 - 2iu\sigma^2)\right]$$

$$\cdot \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{c^2}{4}\right)^k \sum_{r=0}^{k} \frac{(2\sigma^2/c^2)^r}{(k-r)!\ (1 - 2iu\sigma^2)^{r+1}}$$

from which the characteristic function of $\Gamma(j)$ is obtained as:

$$\Phi_\Gamma(u) = \left[\Phi(u)\right]^j$$

This can be inverted directly to obtain the probability density function of $\Gamma(j)$:

$$P(\gamma) = \sum_{n=0}^{j} \binom{j}{n} \left[1 - Q(\sigma, c/\sigma)\right]^{j-n} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty}$$

$$\frac{(c^2/4)^{k_1+\cdots+k_n}}{k_1!\ \ldots\ k_n!\,(2\sigma^2)^n} \sum_{r_1=0}^{k_1} \cdots \sum_{r_n=0}^{k_n} \frac{c^{-2(r_1+\ldots+r_n)}\ \exp - n\sigma^2/2}{(k_1-r_1)!\ \ldots(k_n-r_n)!\ (r_1+\ldots+r_n+n-1)!}$$

$$\left(\exp - \frac{\gamma}{2\sigma^2}\right)\ (\gamma - n\,c^2)^{r_1+\ldots+r_n+n-1}, \quad \gamma > n\,c^2$$

Then:

$$P_D(j) = \sum_{n=0}^{j} \binom{j}{n} \left[1 - Q(\sigma, c/\sigma) \right] \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty}$$

$$\frac{(c^2/4)^{k_1 + \cdots + k_n}}{k_1! \cdots k_n!} \sum_{r_1=0}^{k_1} \cdots \sum_{r_n=0}^{k_n} \frac{(2\sigma^2/c^2)^{r_1+\cdots+r_n} \exp - \frac{n}{2}(\sigma^2 + c^2/\sigma^2)}{(k_1-r_1)! \cdots (k_n-r_n)!}$$

$$\left[1 - I\left(\frac{\tilde{C} - n c^2}{2\sigma^2 \sqrt{r_1 + \cdots + r_n + n}} , r_1 + \cdots + r_n + n - 1 \right) \right] \qquad (25)$$

where $\tilde{C} = \max \left\{ C, n c^2 \right\}$. The probability of detection, $P_D(j)$, is plotted vs. the average number of false alarms, $\bar{N}_{fa}(j)$, with c as a parameter in Figure 6.

15. Performance of Threshold-Square Law Processor Under UVTA Conditions

The average number of false alarms under these conditions is given by Equation (24). The single ping processor output with target present under UVTA conditions is denoted $\tilde{X}$ and has probability density function (see Equation (20)):

$$p(\tilde{X}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta(m,n) \, \tilde{X}^{2n+1} \exp - \frac{\tilde{X}^2}{2\sigma^2}$$

Let $\hat{X}$ be the output of a thresholding device whose input is $\tilde{X}$. The probability density function of $\hat{X}^2 = \xi$ is obtained from Equation (22) and is given by:

$$p(\xi) = P_r\left\{\tilde{X} < c\right\} \delta(\xi)$$

$$+ \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta(m,n)\, \xi^n \exp\left(-\frac{\xi}{2\sigma^2}\right) u(\xi - c^2)$$

The characteristic function of $\hat{X}^2$ is then:

$$\Phi(u) = E \exp(iu\hat{X}^2) = \int_0^{\infty} \exp(iu\xi)\, p(\xi)\, d\xi$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta(m,n)}{2}\, n!\, (2\sigma^2)^{n+1}\, I\left(\frac{c^2/2\sigma^2}{\sqrt{n+1}}, n\right)$$

$$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta(m,n)}{2} \exp\left(\frac{2iu\sigma^2 c^2 - c^2}{2\sigma^2}\right)$$

$$\cdot \sum_{r=0}^{n} \frac{n!}{(n-r)!} \frac{(c^2)^{n-r}}{\left(-iu + \frac{1}{2\sigma^2}\right)^{r+1}}$$

The characteristic function of $\Gamma(j)$ will then be:

$$\Phi_{\Gamma}(u) = \left[\Phi(u)\right]^j$$

which can be inverted using Pair 431 of Reference 5 to yield:

$$p(\gamma) = \sum_{k=0}^{j} \binom{j}{k} \left[\lambda(c)\right]^{j-k} \sum_{m_1=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \sum_{n_k=0}^{\infty}$$

$$\frac{\beta(m_1, n_1) \ldots \beta(m_k, n_k)}{2^k} \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \theta(n_1, r_1) \ldots \theta(n_k, r_k)$$

$$\frac{1}{(r_1 + \ldots + r_k + k-1)!} (\gamma - k c^2)^{r_1 + \ldots + r_k + k-1} \exp\left(-\frac{\gamma - k c^2}{2\sigma^2}\right), \quad \gamma > k c^2$$

where:

$$\lambda(c) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta(m,n)\, I\left(\frac{c^2}{2\sigma^2\sqrt{n+1}}, n\right),$$

$$\theta(n,r) = \frac{n!}{(n-r)!} (c^2)^{n-r} \exp -\frac{c^2}{2\sigma^2}$$

$$\delta(m,n) = 2^n (\sigma^2)^{n+1} (n!)\, \beta(m,n)$$

and $\beta(m,n)$ is defined in Section 13.

Then:

$$P_D(j) = \int_C^{\infty} p(\gamma)\, d\gamma$$

$$= \sum_{k=0}^{j} \binom{j}{k} \left[\lambda(c)\right]^{j-k} \sum_{m_1=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \sum_{n_k=0}^{\infty}$$

$$\frac{\beta(m_1, n_1) \ldots \beta(m_k, n_k)}{2^k} \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \theta(n_1, r_1) \ldots \theta(n_k, r_k)$$

$$(2\sigma^2)^{r_1 + \ldots + r_k + k} \left[1 - I\left(\frac{\tilde{C}_4 - k c^2}{2\sigma^2 \sqrt{r_1 + \ldots + r_k + k}}, r_1 + \ldots + r_k + k - 1 \right) \right] \quad (26)$$

where:

$\tilde{C}_4 = \max \left\{ C_4, k c^2 \right\}$. The probability of detection, $P_D(j)$, is plotted vs. the average number of false alarms in Figure 7 for $\sigma^2 = 1$ and in Figure 8 for $\sigma^2 = 10$.

16. Probability of Detection with Data Reduction Scheme

The test function $\Gamma(j)$ is to be computed for the path extensions corresponding to the largest values of X and the test functions for other path extensions are to be dumped. The new probability of detection $\hat{P}_D(j)$ becomes:

$$\hat{P}_D(j) = P_r\left\{\text{detection after j pings} \,\middle|\, \text{true target path is not dumped}\right\}$$

$$\cdot\, P_r\left\{\text{true target path is not dumped}\right\}$$

$$= P_D(j) \cdot P_r\left\{\text{true target path is not dumped}\right\}$$

Now:

$$P_r\left\{\text{true target path is not dumped}\right\}$$

$$= \prod_{k=2}^{j} P_r\left\{\text{true target path is not dumped on } k^{th} \text{ ping}\right\}$$

$$= \left[1 - P_r\left\{\mu \text{ or more of H-1 independent Rayleigh random variables exceed one Rice random variable}\right\}\right]^{j-1}$$

$$= \left[1 - \sum_{k=\mu}^{H-1} \binom{H-1}{k} \left(P_r\left\{X_o > X_1\right\}\right)^k \left(1 - P_r\left\{X_o > X_1\right\}\right)^{H-k-1}\right]^{j-1} \qquad (27)$$

where X_o has a Rayleigh distribution and X_1 has a Rice distribution:

$$p_o(X_o) = \frac{X_o}{\sigma^2} \exp - \frac{X_o^2}{2\sigma^2}$$

$$p_1(X_1) = \frac{X_1}{\sigma^2} \exp\left(- \frac{X_1^2 + \sigma^4}{2\sigma^2}\right) I_o(X_1) \tag{28}$$

Let $F_o(\cdot)$ denote the distribution function of X_o:

$$F_o(x) = P_r\left\{X_o \leq x\right\} = \int_o^x p_o(\tau)\, d\tau$$

$$= 1 - \exp - \frac{x^2}{2\sigma^2}$$

Conditioning on X_1 we have:

$$P_r\left\{X_o > X_1\right\} = \int_o^\infty P_r\left\{X_o > X_1 \middle| X_1 = x\right\} p_1(x)\, dx$$

$$= \int_o^\infty \left[1 - F_o(x)\right] p_1(x)\, dx$$

$$= \int_o^\infty \left[\exp - \frac{x^2}{2\sigma^2}\right]\left[\frac{x}{\sigma^2} \exp\left(- \frac{x^2 + \sigma^4}{2\sigma^2}\right) I_o(x)\right] dx$$

$$= \frac{1}{2} \exp - \frac{\sigma^2}{4}$$

Substituting in (27) we obtain:

P_r {true target path is not dumped}

$$= \left[1 - \sum_{k=\mu}^{H-1} \binom{H-1}{k} \left(\tfrac{1}{2} \exp - \frac{\sigma^2}{4}\right)^k \left(1 - \exp - \frac{\sigma^2}{4}\right)^{H-k-1} \right]^{j-1} = \frac{\hat{P}_D(j)}{P_D(j)}$$

Under UVTA conditions, X is replaced by $\tilde{X}$ and $p_1(x)$ of Equation (28) is replaced by (see Equation (14)):

$$p_1(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta(m,n) \; x^{2n+1} \; \exp - \frac{x^2}{2\sigma^2}$$

Then:

$$P_r \left\{ X_o > \tilde{X}_1 \right\} = \int_0^{\infty} \left[1 - F_o(x) \right] \; \tilde{p}_1(x) \, dx$$

$$= \int_0^{\infty} \left(\exp - \frac{x^2}{2\sigma^2}\right) \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta(m,n) \, x^{2n+1} \exp - \frac{x^2}{2\sigma^2}\right) dx$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m \, (\sigma^2)^{m+n} \, (1;\, 2;\, m+n)}{m! \; n! \; (m+n)! \; 2^{2m+3n+1}} = p_f \qquad (29)$$

Substituting (29) in (27) we obtain:

P_r {true target is not dumped under UVTA conditions}

$$= \left[1 - \sum_{k=\mu}^{H-1} \binom{H-1}{k} p_f^k \, (1 - p_f)^{H-k-1} \right]^{j-1}$$

$$= \frac{\hat{P}_D(j)}{P_D(j)}$$

The degradation factor $\tilde{P}_D(j)/P_D(j)$ is plotted vs. μ/H with σ^2 as a parameter for additive noise conditions in Figure 9 and for UVTA conditions in Figure 10.

REFERENCES

1. Marcum, J.I., "A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix", RM-753, The Rand Corp., 25 April 1952.

2. Woodward, P.M., Probability and Information Theory with Applications to Radar, Pergamon Press, New York, 1953.

3. Technical Memo, Active Data Processor Requirements for Current Baseline System, Conformal/Planar Array Sonar Project, C026-48009, GE-HMED, 10 August 1966.

4. Omura, J. and Kailath, T., "Some Useful Probability Distributions", Stanford University Technical Report #7050-6, September 1965.

5. Campbell, G. and Foster, R., Fourier Integrals for Practical Applications, D. Van Nostrand Co., New York, 1948.

6. Abramson, N. and Farison, J., "On Applied Decision Theory", Stanford University Technical Report #2005-2, September 1962.

7. Wax, N., Selected Papers on Noise and Stochastic Processes, Dover Publications, New York, 1954.

8. Technical Memo, Active Data Processing Report for Current Baseline System (U), C/PAS, C026-48006, GE-HMED, 1 April 1966, (Confidential).